Application of Finite Difference Method for the Analysis of a Rectangular Thin Plate with Eccentric Opening

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Abstract

It is a common practice to use steel plates for supporting lateral or vertical loads. A number of analytical and numerical methods are developed for the analysis of such plates. In this work, a rectangular steel plate subjected to uniformly distributed loading with all around simply supported edges having an eccentric rectangular opening is analyzed using the finite difference method, a straightforward numerical approach. The work covers the determination of displacement components at different points and comparing the result with identically simulated result obtained from conventional software package ETABS. The deflection values calculated by finite difference method were found to vary about 8% with that obtained from ETABS simulation. Location of maximum deflection was the same for both case and its value varied by 3% only. Deflection pattern was the same in both the cases which indicates that the finite difference method holds good for the analysis of thin plates.
Keywords: *thin plate analysis, finite difference method, hollow plate.*

1 Introduction

The use of plates and plated structural element is very common in modern steel structures. In many cases these plates are subjected to lateral loads. Analysis and design of such plates are performed using various classical, numerical and engineering techniques. To analyze a thin plate one needs to solve its established relations between loads, deflections, bending, stress etc. The solution method may be either analytical or numerical. The analytical methods are the old ones and they imply determination of exact mathematical functions defining the solutions in a closed form. On the other hand, in the numerical methods, the resulting equations including boundary conditions are solved in approximate numerical way. A numerical method is often preferred as it can be incorporated with computers easily. In this paper "Finite Difference Method", a convenient and straightforward numerical approach is used for the analysis of a thin plate. The proposed method can be easily programmed to readily apply on a plate problem.

2 Classical Plate Theory and Governing Equation

An exact stress analysis of a thin plate subjected to loads acting normal to its surface requires solution of the differential equations of three-dimensional elasticity. The exact bending analysis for thin plates (Timoshenko and Woinowsky-Krieger, 1959) is somewhat complex and time-consuming. To avoid mathematical difficulties associated with this exact equation, the Kirchhoff's classical theory of thin plates is used with sufficient accuracy in results without the need of carrying out a full three-dimensional stress analysis. The Kirchhoff's theory is expressed in terms of transverse deflections $W(x,y)$ for which the governing differential equation is of fourth order, requiring only two boundary conditions to be satisfied at each edge. The governing differential equation for deflection of thin plates by Kirchhoff's theory can be presented as (Ugural, 1999)

$$\nabla^4 W(x,y) = \frac{P_z(x,y)}{D} \qquad (1)$$

Where,
x,y = Coordinate of pont on the plate
W = Deflection in z direction
P_z = Applied uniformly distributed static load in z direction

∇^4 =Biharmonic differential operator= $\nabla^2\ \nabla^2$
D = Flexural rigidity
Expanding the biharmonic operator the equation (1) can be simplified as (Szilard, 2004)

$$\frac{\delta^4 W}{\delta x^4} + 2\frac{\delta^4 W}{\delta x^2 \delta y^2} + \frac{\delta^4 W}{\delta y^4} = \frac{P_z(x,y)}{D} \qquad (2)$$

The flexural rigidity can be defined by the expression (Ghods and Mir, 2014),

$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad (3)$$

In this expression,
E = Modulus of elasticity plate material
h = Thickness of plate
μ = Poisson's ratio for plate material
The equation (2) is readily solvable using finite difference method.

3 The Finite Difference Method

The finite difference method, usually referred as FDM, is a numerical method that solves the equations of boundary problems using mathematical discretization. It is the oldest but still very viable numerical methods for solution of partial differential equation and hence is suitable for solving plate bending equation. This method is sufficiently accurate for thin plate analysis (Ezeh et al., 2013). It is probably the most transparent and the most general method among the various numerical approaches available for analysis of thin plates. Although, a number of finite element based software packages are available now-a-days, the finite difference method is still regarded for its straightforward nature and a minimum requirement on hardware.

3.1 FDM Expressions and Their Graphical Presentation

To apply the finite difference method, some imaginary nodes or joints are to be considered on the plate to be analyzed. The derivatives in the governing differential equations are then replaced by difference quantities at those nodes. The nodes can be located at the joints of some imaginary square, rectangular, triangular or other reference network, called a finite difference mesh. Doing so, a set of algebraic equations are formed from which the plate deflections at the nodes can be obtained.
Figure 1 shows a plate with deflection elements at different nodes.

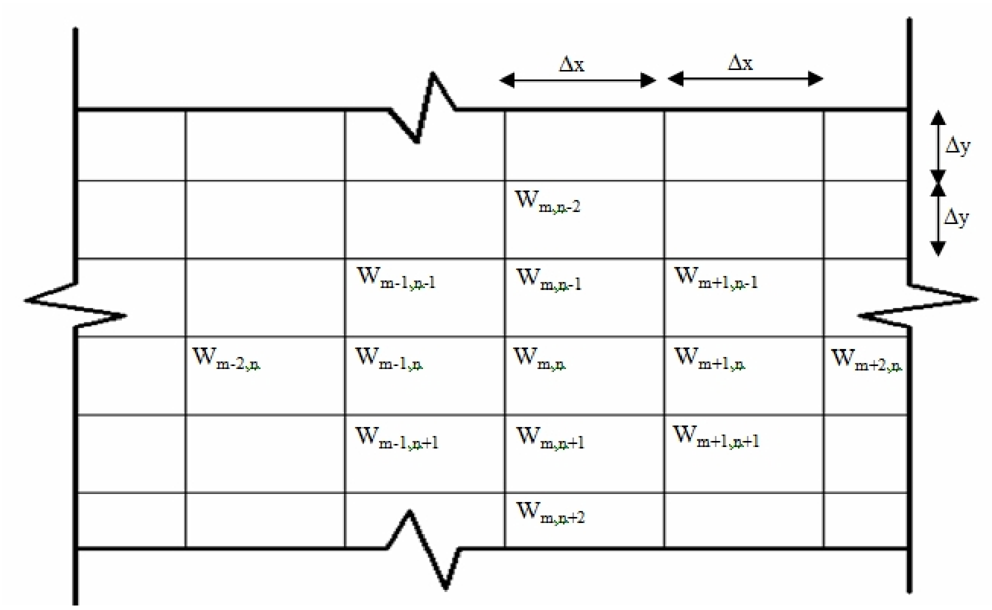


Figure 1. Plate element with nodes and deflection components

The FDM expressions required to solve the equation (2) at a node (m,n) in figure 1 are (Szilard, 2004),

$$\frac{\delta^2 W}{\delta x^2} = \frac{1}{(\Delta x)^2}\left(W_{m+1,n} - 2W_{m,n} + W_{m-1,n}\right)$$

$$\frac{\delta^2 W}{\delta y^2} = \frac{1}{(\Delta y)^2}\left(W_{m,n+1} - 2W_{m,n} + W_{m,n-1}\right)$$

$$\frac{\delta^4 W}{\delta x^4} = \frac{1}{(\Delta x)^4}\left(W_{m+2,n} - 4W_{m+1,n} + 6W_{m,n} - 4W_{m-1,n} + W_{m-2,n}\right)$$

$$\frac{\delta^4 W}{\delta y^4} = \frac{1}{(\Delta y)^4}\left(W_{m,n+2} - 4W_{m,n+1} + 6W_{m,n} - 4W_{m,n-1} + W_{m,n-2}\right)$$

Here Δx is the spacing of nodes along the direction of m and Δy is the spacing of nodes along the direction of n. If the nodes are equally spaced in both direction, then spacing, $\Delta = \Delta x = \Delta y$
Replacing the derivatives in equation (2), for the node (m,n) we get,

$$\begin{aligned} &20W_{m,n} - 8\left(W_{m+1,n} + W_{m-1,n} + W_{m,n+1} + W_{m,n-1}\right) \\ &+ 2\left(W_{m+1,n+1} + W_{m-1,n+1} + W_{m+1,n-1} + W_{m-1,n-1}\right) \\ &+ W_{m+2,n} + W_{m-2,n} + W_{m,n+2} + W_{m,n-2} = \frac{\Delta^4 P_z}{D} \end{aligned} \quad (4)$$

This expression is presented graphically in figure 2(a) (Szilard, 2004)

3.2 Treatment of Boundary Condition

To make use of equation (4) or its corresponding graphical stencil, the point (m,n) must be accompanied with 12 surrounding points. This is possible for internal points of a plate. But on or near the edges of a plate all the 12 surrounding points are not available. In that case some fictitious noted are imagined outside the plate boundary and proper boundary conditions are applied to express these fictitious deflections in terms of the deflection components inside the plate.
The conditions to be applied for edges are: (Szilard,2004)
For a point (m,n) on fixed edge: $W_{m,n} = 0$ and $W_{m+1,n} = W_{m-1,n}$
For a point (m,n) on simply supported edge: $W_{m,n} = 0$ and $W_{m+1,n} = -W_{m-1,n}$
For a point (m,n) on free edge requires 4 fictitious points outside the plate boundary. In such case following conditions need to be applied,

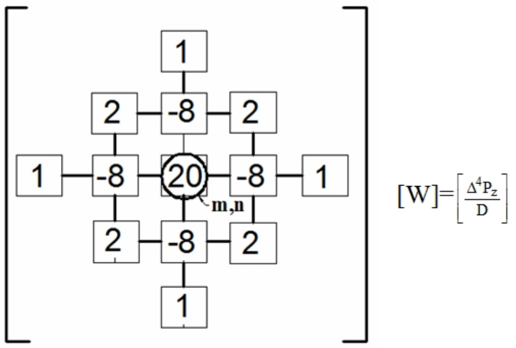
$$W_{m,n+1} = 2(1+\mu)W_{m,n} - W_{m,n-1} - \mu\left(W_{m-1,n} + W_{m+1,n}\right)$$

$$W_{m,n+2} = W_{m,n-2} - (6-2\mu)\left(W_{m,n-1} - W_{m,n+1}\right) - (2-\mu)\left(W_{m+1,n+1} + W_{m-1,n+1} - W_{m-1,n-1} - W_{m+1,n-1}\right)$$

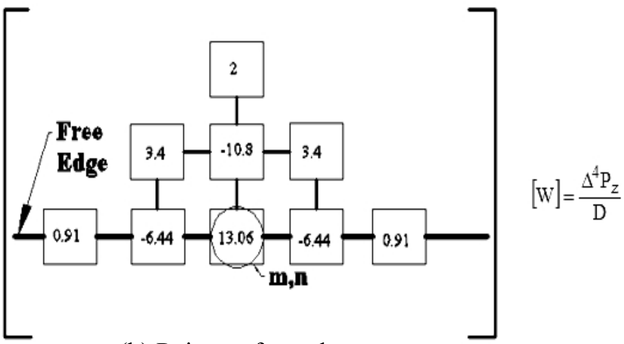
$$W_{m-1,n+1} = 2(1+\mu)W_{m-1,n} - W_{m-1,n-1} - \mu\left(W_{m-2,n} + W_{m,n}\right)$$

$$W_{m+1,n+1} = 2(1+\mu)W_{m+1,n} - W_{m+1,n-1} - \mu\left(W_{m+2,n} + W_{m,n}\right)$$

By applying the boundary conditions to FDM expressions for different nodes near the free edge can be expressed graphically for $\mu = 0.3$ (Barton, 1948) as shown in figure 2(b) and figure 2(c)

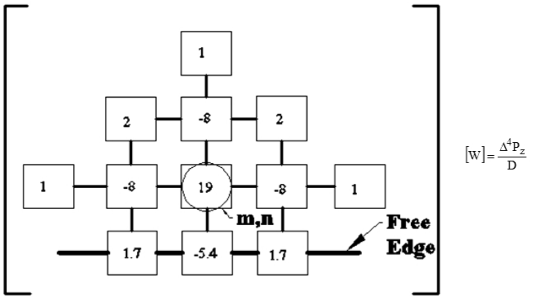


(a) Internal point

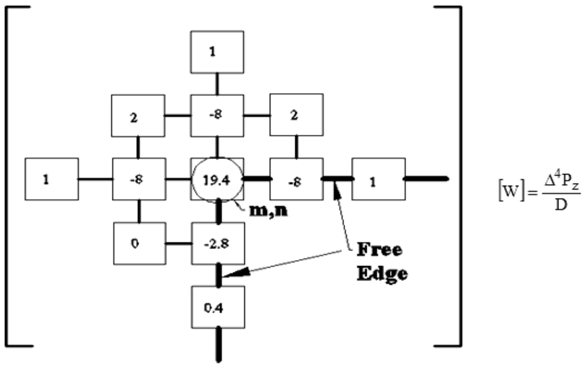


(b) Point on free edge

Figure 2. Graphical presentation of FDM expressions for different node location.



(c) Point adjacent to free edge



(d) Point on the corner of opening

Figure 2. Graphical presentation of FDM expression for different node location (contd.)

In similar manner a graphical presentation for the points at the corner of internal opening is constructed for $\mu = 0.3$ and presented in figure 2(d).

3.3 Solution of FDM expressions by Matrix Method

By applying the FDM expression in equation (4) at all the nodes of the plate, a system of linear equations will be formed. Number of equations will be evidently equal to the number of point deflections and therefore, the system can be solved and deflections at all the nodes can be calculated out. Solution of the system by matrix method seems to be favorable as the system involves a large number of unknowns. To do so, a coefficient matrix needs to be formed and it is to be inversed. The inverse matrix is then multiplied to the right hand side quantity in equation (4) to get the solution matrix.

4 Experimental Plate Analyses

A thin rectangular plate with an internal opening is considered for an experimental analysis by finite difference method. To avoid complicacy, boundary conditions are chosen to be simply supported on all edges. Matrix method is used for solving the system of equations.

4.1 Specification and Initial Calculations

The experimental analysis is made considering the following specifications:

Length of plate = 4 m
Width of plate = 3 m
Thickness of plate = 100 mm
Young's modulus of plate material = 200 GPa
Poisson's ratio of materials = 0.30
Uniformly distributed superimposed load = 10 kPa

Flexural rigidity of plate is calculated using equation (3) as:

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} = \frac{200\times10^3\times100^3}{12\left(1-0.30^2\right)} = 1.832\times10^{10}\,\mathrm{N-mm}$$

Spacing on both direction is considered to be 500mm and thus, right hand side of the equation (4) is calculated as

$$\frac{\Delta^4 P_z}{D} = \frac{500^4\times10\times10^{-3}}{1.832\times10^{10}} = 0.0341\mathrm{mm}$$

4.2 Node Generation

The nodes are generated by dividing the plate into some square meshes. The length is divided into 8 equal parts and the width in 6 equal parts resulting in $500\mathrm{mm}\times500\mathrm{mm}$ square meshes as shown in figure 3. From structural symmetry point of view, there are some points with identical loading and boundary conditions. Taking it into consideration, total 20 numbers of unknown node deflections are found and they are designated as W_1, W_2, W_3 ….. etc. At the nodes along the outer edges deflections are marked as zero. To treat the boundary conditions some fictitious deflections are also shown outside the simply supported edge.

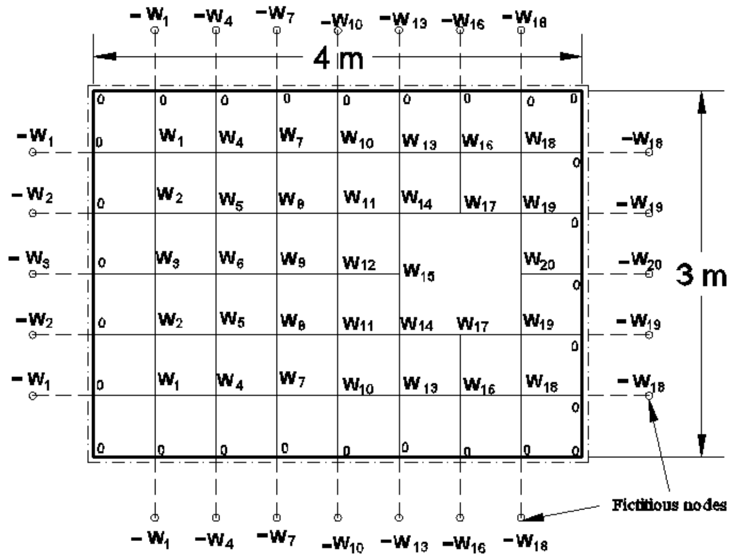


Figure 3. Experimental plate with unknown node displacement

4.3 Application of FDM Expressions

Studying the node positions, appropriate stencils to be used are identified as tabulated in table 1.

Table 1. Selection of stencil for different nodes

Node location	Deflection components	Applicable Stencil
Internal points	W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{13} and W_{18}	Figure 2 (a)
Points on free edges	W_{15}, W_{17} and W_{20}	Figure 2 (b)
Points Adjacent to free edges	W_{12} and W_{16}	Figure 2 (c)
Points at the corners of internal opening	W_{14} and W_{19}	Figure 2 (d)

The equations obtained by applying FDM expressions in 20 nodes are tabulated in table 2

Table 2. Equations at different nodes

Node	Equation
W_1	$18W_1-8W_2+W_3-8W_4+2W_5+W_7= 0.0341$
W_2	$-8W_1+20W_2-8W_3+2W_4-8W_5+2W_6+ W_8= 0.0341$
W_3	$2W_1-16W_2+19W_3+4W_5-8W_6+W_9= 0.0341$
W_4	$-8W_1+2W_2+19W_4-8W_5+W_6-8W_7+2W_8+W_{10}= 0.0341$
W_5	$2W_1-8W_2+2W_3-8W_4+21W_5-8W_6+2W_7-8W_8+2W_9+W_{11}= 0.0341$
W_6	$4W_2-8W_3+2W_4-16W_5+20W_6+4W_8-8W_9+W_{12}= 0.0341$
W_7	$W_1-8W_4+2W_5+19W_7-8W_8+W_9-8W_{10}+2W_{11}+W_{13}= 0.0341$
W_8	$W_2+2W_4-8W_5+2W_6-8W_7+21W_8-8W_9+2W_{10}-8W_{11}+2W_{12}+W_{14}= 0.0341$
W_9	$W_3+4W_5-8W_6+2W_7-16W_8+20W_9+4W_{11}-8W_{12}+W_{15}= 0.0341$
W_{10}	$W_4-8W_7+2W_8+19W_{10}-8W_{11}+W_{12}-8W_{13}+2W_{14}+W_{16}= 0.0341$
W_{11}	$W_5+2W_7-8W_8+2W_9-8W_{10}+21W_{11}-8W_{12}+2W_{13}-8W_{14}+2W_{15}+W_{17}= 0.0341$
W_{12}	$W_6+4W_8-8W_9+2W_{10}-16W_{11}+19W_{12}+3.4W_{14}-5.4W_{15}= 0.0341$
W_{13}	$W_7-8W_{10}+2W_{11}+19W_{13}-8W_{14}+W_{15}-8W_{16}+2W_{17}+W_{18}= 0.0341$
W_{14}	$W_8+2W_{10}-8W_{11}-8W_{13}+19.8W_{14}-2.8W_{15}+2W_{16}-8W_{17}+W_{19}= 0.0341$
W_{15}	$2W_9+6.8W_{11}-10.8W_{12}+1.82W_{13}-12.88W_{14}+13.06W_{15}= 0.0341$
W_{16}	$W_{10}-8W_{13}+1.7W_{14}+18W_{16}-5.4W_{17}-8W_{18}+1.7W_{19}= 0.0341$
W_{17}	$0.91W_{11}+3.4W_{13}-6.44W_{14}-10.8W_{16}+13.06W_{17}+3.4W_{18}-6.44W_{19}= 0.0341$
W_{18}	$W_{13}-8W_{16}+2W_{17}+18W_{18}-8W_{19}+W_{20}= 0.0341$
W_{19}	$W_{14}+2W_{16}-8W_{17}-8W_{18}+18.8W_{19}-2.8W_{20}= 0.0341$
W_{20}	$1.82W_{18}-12.88W_{19}+11.06W_{20}= 0.0341$

This system of equations in table 2 is readily solvable using matrix method.

5 Result and Comparison

The equations presented in table 2 is solved using matrix method and resulting deflections are compared with software package ETABS (ETABS 9.7.3, 2011) analysis result. The comparison is presented in table 3.

Table 3. Result and comparison

Node deflection	FDM analysis result (mm)	ETABS analysis result (mm)	Difference	
			mm	%
W_1	0.0669	0.0613	-0.0056	9.14
W_2	0.1128	0.1038	-0.009	8.67
W_3	0.1288	0.1187	-0.0101	8.51
W_4	0.1196	0.1095	-0.0101	9.22
W_5	0.2026	0.1865	-0.0161	8.63
W_6	0.2317	0.2137	-0.018	8.42
W_7	0.1542	0.1407	-0.0135	9.59
W_8	0.2625	0.2407	-0.0218	9.06
W_9	0.3003	0.2764	-0.0239	8.65
W_{10}	0.1697	0.1551	-0.0146	9.41
W_{11}	0.2918	0.2685	-0.0233	8.68
W_{12}	0.3345	0.3095	-0.025	8.08
W_{13}	0.1629	0.1498	-0.0131	8.74
W_{14}	0.2884	0.2753	-0.0131	4.76
W_{15}	0.3431	0.3331	-0.01	3
W_{16}	0.1242	0.1206	-0.0036	2.99
W_{17}	0.2247	0.2321	0.0074	-3.19
W_{18}	0.0679	0.0693	0.0014	-2.02
W_{19}	0.1168	0.1284	0.0116	-9.03
W_{20}	0.128	0.1474	0.0194	-13.16

6 Discussion on Results and Conclusions

The data presented in table 3 indicate a considerable accuracy of finite difference method for the analysis of thin plates. Deflection values calculated by FDM at different nodes are found to vary about 8% with more accurate result obtained by ETABS analysis. It is notable that, the deflection pattern is similar for both the analyses. In both case maximum deflection occurs at the node W_{15} and its values are found to be 0.3431 mm (FDM) and 0.3331 mm (ETABS) respectively. The maximum deflection obtained by FDM analysis is increased by only 3%. The finite difference method has its advantage of being straight forward in nature. Any other boundary condition such as clamped or free can also be incorporated with this method without any complexity. The variation found in the example analysis is very small and reasonable. This method was previously tested with STAAD.Pro analysis (Roknuzzaman et al., 2015) and also found to hold good. However, more accuracy in result can be obtained considering finer meshes or by using improved finite difference method (IFDM).

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